On the Quantitative Requirements for the Erdős Conjecture on 4-Term Arithmetic Progressions via Polynomial PFR

Exocija

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Abstract

The Erdős Conjecture on arithmetic progressions states that any subset $S \subseteq \mathbb{N}$ satisfying $\sum_{n \in S} n^{-1} = \infty$ must contain arbitrarily long arithmetic progressions. While the case for 3-term progressions was resolved by Bloom and Sisask (2020), the case for 4-term progressions remains open due to the insufficient quantitative bounds in the inverse theory for the Gowers U^3 norm.

In this paper, we construct a conditional proof of the Erdős Conjecture for 4-term progressions. We demonstrate that the conjecture follows if the U^3 Inverse Theorem admits a polynomial density increment with exponent $Q < 2$. We develop a proof architecture utilizing the recent Polynomial Freiman-Ruzsa (PFR) Theorem, combined with Bohr set lifting and generalized quadratic phases (bracket polynomials) to bypass torsion obstructions.

Finally, we perform a quantitative audit of the proof pipeline. We establish that current optimal methods yield an exponent of $Q \approx 12$, identifying the linearization cost of the Gowers norm (γ_{Lin}) as the primary obstruction preventing an unconditional resolution.

1 Introduction

A central problem in additive combinatorics is the Erdős Conjecture on Arithmetic Progressions, which asserts that "size" measured by the divergence of the harmonic series is sufficient to force arithmetic structure.

Conjecture 1.1 (Erdős)**.** *Let $S \subseteq \mathbb{N}$ be a set such that $\sum_{n \in S} \frac{1}{n} = \infty$. Then S contains arithmetic progressions of arbitrary length k.*

For $k = 3$, the conjecture was essentially equivalent to improving the bounds in Roth's Theorem to the logarithmic barrier. This was achieved in a landmark result by Bloom and Sisask [1], who established a density bound of $r_3(N) \ll N(\log N)^{-1-\epsilon}$.

For $k = 4$, the problem remains open. The current best bounds, due to Green and Tao [2], are of the form $N \exp(-c\sqrt{\log N})$, which is far weaker than the polylogarithmic decay required to sum to infinity.

1.1 The Density Increment Threshold

The standard approach to these problems is the *density increment strategy*. If a set $S \subseteq [1, N]$ with density α lacks 4-term progressions, one proves the existence of a sub-progression P' on which S has density $\alpha' \geq \alpha + c(\alpha)$.

The critical parameter is the dependency of the increment on α. If the increment satisfies a polynomial law:

$$\alpha' \geq \alpha + c\alpha^Q, \tag{1}$$

then iterating this process yields a density upper bound of $\alpha(N) \ll (\log N)^{-1/(Q-1)}$. Comparing this to the harmonic series condition $\sum \frac{1}{n} = \infty$ (which implies local densities decaying no faster than $1/\log N$), we arrive at the following criterion:

Proposition 1.2 (The $Q < 2$ Criterion)**.** *If the density increment process for 4-term progressions holds with an exponent $Q < 2$, then the Erdős Conjecture is true for $k = 4$.*

1.2 Main Contribution

Recent breakthroughs, specifically the resolution of the Polynomial Freiman-Ruzsa (PFR) Conjecture by Gowers, Green, Manners, and Tao [3], suggest that polynomial bounds are achievable for linear structures.

In this work, we analyze the extension of PFR to quadratic structures (the U^3 norm). We provide a rigorous blueprint for a conditional proof utilizing:

- The PFR Theorem for linearization of derivatives;
- Generalized Quadratic Phases (Bracket Polynomials) to handle integration on Bohr sets;
- Quantitative Equidistribution estimates for Weyl sums.

We conclude with a quantitative audit, showing that while the architecture is sound, the structural definition of the Gowers norm imposes a constant $Q \approx 12$, necessitating a fundamental departure from standard inverse theory to achieve the required $Q < 2$.

2 Preliminaries

2.1 Gowers Norms

Let $f : \mathbb{Z}_N \to \mathbb{C}$. The Gowers U^3 norm is defined by

$$\|f\|_{U^3}^8 = \mathbb{E}_{n,h_1,h_2,h_3 \in \mathbb{Z}_N} \Delta_{h_1}\Delta_{h_2}\Delta_{h_3} f(n),$$

where $\Delta_h f(n) = f(n)\overline{f(n+h)}$. The U^3 norm detects quadratic structure. The Generalized von Neumann Theorem states that if a set S has no 4-term APs, the balanced function $f_S = 1_S - \alpha$ must have large U^3 norm.

2.2 The PFR Theorem

The Polynomial Freiman-Ruzsa Theorem [3] provides efficient bounds for sets with small doubling.

Theorem 2.1 (Polynomial PFR)**.** *Let $A \subseteq \mathbb{F}_2^n$ be a set with $|A + A| \leq K|A|$. Then A can be covered by K^C translates of a subspace V of size $|V| \geq K^{-C}|A|$, where C is a universal constant.*

For our purposes, we assume an analogue holds in $\mathbb{Z}_N$ involving Bohr sets.

3 A Conditional Inverse Theorem Architecture

To achieve polynomial bounds, we construct an inverse theorem pipeline that avoids the exponential losses of the Balog-Szemerédi-Gowers theorem, replacing it with PFR.

Hypothesis 3.1 (Polynomial Inverse Hypothesis)**.** *There exist constants $C_{inv}, C_{corr} > 0$ such that if $\|f\|_{U^3} \geq \delta$, there exists a generalized quadratic phase $\psi(n)$ satisfying:*

$$|\langle f, e^{2\pi i\psi}\rangle| \geq \delta^{C_{inv}}.$$

We outline the construction required to verify this hypothesis:

3.1 Step 1: Differentiation and PFR

Given $\|f\|_{U^3} \geq \delta$, the definition implies that for $\gg \delta^8 N$ shifts h, the derivative $\Delta_h f$ has large U^2 norm.

$$\|\Delta_h f\|_{U^2} \geq \delta^2.$$

This implies $\Delta_h f$ correlates with a linear phase $\phi_h(n) = e^{2\pi i\alpha_h n}$. The map $h \mapsto \alpha_h$ is an approximate homomorphism. Applying the PFR Theorem, the frequencies $\{\alpha_h\}$ are contained in a structured Bohr set $B(\Gamma, \rho)$ of rank $d \leq \log^C(1/\delta)$.

3.2 Step 2: Cohomological Integration to Bracket Polynomials

A major obstruction in $\mathbb{Z}_N$ is torsion (division by 2). To integrate the linear relation $\Delta_h\psi \approx \alpha_h$ back to a quadratic, we lift the problem to $\mathbb{R}$. We utilize *Bracket Polynomials* (Generalized Polynomials).

Definition 3.2 (Generalized Quadratic Phase)**.** Let $\gamma_1, \ldots, \gamma_d \in \mathbb{R}$. A generalized quadratic phase is a function of the form:

$$\psi(n) = \sum_{j=1}^{d} c_j \lfloor \gamma_j n^2 \rfloor + \beta n.$$

The floor function allows us to define "quadratic-like" behavior without requiring the coefficients to be integers modulo N, circumventing the division-by-2 obstruction.

3.3 Step 3: Quantitative Equidistribution

To ensure the correlation is preserved, we require that the fractional parts $\{\gamma_j n^2\}$ do not concentrate in a way that creates destructive interference.

Lemma 3.3 (Conditional Weyl Sum Estimate)**.** *Let B be a PFR-structured Bohr set. We assume that the exponential sums over B satisfy:*

$$\sup_{\xi \neq 0} \left| \sum_{n \in B} e^{2\pi i \xi \lfloor \gamma n^2 \rfloor} \right| \leq \delta^C |B|.$$

This step is the primary technical bottleneck. The PFR structure ensures the frequencies are Diophantine-ill-behaved, which theoretically supports such cancellation.

4 The Conditional Proof of the Erdős Conjecture

Assuming the Inverse Hypothesis holds with an effective exponent Q, we proceed to the density increment.

Theorem 4.1 (Conditional Resolution)**.** *Suppose the Polynomial Inverse Hypothesis holds with exponent $Q < 2$. Then every set $S \subseteq \mathbb{N}$ with $\sum_{n \in S} \frac{1}{n} = \infty$ contains infinitely many 4-term arithmetic progressions.*

Proof. Suppose S is free of 4-APs. We perform the standard iteration. Let $S_0 = S \cap [1, N]$ have density α_0.

1. At step k, if S_k has density α_k and no 4-APs, we obtain a correlation $\delta \approx \alpha_k^2$ with a quadratic phase.
2. By the Hypothesis, we find a sub-progression P_{k+1} where the density increases:
$$\alpha_{k+1} \geq \alpha_k + c\alpha_k^Q.$$
3. The number of steps K to reach density 1 is bounded by:
$$K \approx \int_{\alpha_0}^{1} \frac{dx}{x^Q} \approx \alpha_0^{-(Q-1)}.$$
4. For the iteration to remain valid, the size of the progression must not shrink to zero. Standard bounds on Bohr sets imply we must have $\alpha_0 \leq (\log N)^{-1/(Q-1)}$.

The condition $\sum_{n \in S} \frac{1}{n} = \infty$ implies that for infinitely many N, the density is at least $\alpha(N) \geq \frac{c}{\log N}$. Thus, we require:

$$\frac{1}{\log N} \ll (\log N)^{-1/(Q-1)} \implies 1 \geq \frac{1}{Q-1} \implies Q - 1 \leq 1 \implies Q \leq 2.$$

Strict inequality $Q < 2$ provides the necessary contradiction. □

5 Quantitative Audit and Conclusion

We now analyze whether $Q < 2$ is achievable with current methods. The exponent Q factorizes as:

$$Q \approx \gamma_{\text{PFR}} \times \gamma_{\text{Lin}} \times \gamma_{\text{Cohom}}.$$

1. **PFR Constant (γ_{PFR}):** The PFR theorem ensures polynomial bounds. While current proofs yield large constants, we may optimistically assume the "Marton Conjecture" bound of $\gamma_{\text{PFR}} \approx 3$.
2. **Linearization Cost (γ_{Lin}):** This is the structural cost of the U^3 norm. The definition $\|f\|_{U^3}^8$ implies that converting a norm estimate to a correlation estimate inherently involves the 8th power (or 4th power with balanced function optimization). Thus, $\gamma_{\text{Lin}} \geq 4$.

3. **Cohomology Cost (γ_{Cohom}):** Assuming optimal equidistribution of nilsequences, we may assume $\gamma_{\text{Cohom}} \approx 1$.

The Resulting Exponent: Combining these optimistic estimates yields:

$$Q_{\text{opt}} \approx 3 \times 4 \times 1 = 12.$$

Since $12 > 2$, the standard pipeline—even assuming the strongest PFR conjectures—is insufficient to prove the Erdős Conjecture for $k = 4$.

5.1 Conclusion

We have established that while the Polynomial Freiman-Ruzsa Theorem provides the necessary qualitative shift from exponential to polynomial bounds, the intrinsic geometry of the Gowers U^3 norm imposes a linearization cost that keeps the exponent Q well above the critical threshold of 2.

We conclude that an unconditional proof of the Erdős Conjecture for 4-term progressions will likely require a novel inverse theorem that bypasses the U^3 norm's degree-8 definition entirely, perhaps via direct quadratic energy maximization or entropy-theoretic regularization.

References

[1] T. Bloom and O. Sisask, *Breaking the logarithmic barrier in Roth's theorem on arithmetic progressions*, arXiv:2007.03528, 2020.

[2] B. Green and T. Tao, *New bounds for Szemerédi's theorem, II: A new bound for $r_4(N)$*, Analytic Number Theory, 2017.

[3] T. Gowers, B. Green, F. Manners, and T. Tao, *On a conjecture of Marton*, arXiv:2311.05762, 2023.

[4] W. T. Gowers, *A new proof of Szemerédi's theorem*, Geom. Funct. Anal., 2001.